Lecture 2

Mathematical Models based on 1D ODE

(One dimension Ordinary Differential Equations).

We consider very different processes and will show that their MM belong to the same common class of mathematical problems.

Population dynamics models often using differential equations or matrices, that describe how populations change in size and structure over time. They include factors like birth, death, immigration and emigration. The limiting factors such as environmental limits of carrying capacity.

We restrict ~~to~~ the simple but important models that are based on the assumption that environment is stable and is not changing for time intervals interesting for us in this analysis

In a closed system (where immigration and emigration balance each other) the rate of change in the number of individuals in a population can be described as:

$$\frac{dN}{dt} = B - D,$$

where

N is the ~~total~~ number of individuals

B is the number of births

D is the number of deaths per individual.

In order ~~to~~ to define a closed mathematical model we use a "good" aproximation valid for many practical purposes, that B and ~~D~~ are proportional to the the total number of individuals in the specific population.

$$\frac{dN}{dt} = bN - dN = (b-d)N \qquad (1)$$
$$= rN,$$

where b, d and r stand for the rates of birth, death and the rate of change per individual

In order to simulate~~to~~ the dynamics we should formulate the initial condition

$$N(0) = N_0 \qquad (2)$$

Such problem can be solved in analytical form

$$N(t) = N_0 e^{rt}$$

Let us consider different cases of parameter r

1. $r = 0 \Rightarrow N(t) = N_0, \quad t > 0$

The size of this population is constant and equal to the initial condition. Processes of birth and death are balanced

2. $\boxed{r > 0.}$

$N(t) \longrightarrow \infty$ for $t \to \infty$

The theory developed by
Thomas Robert Malthus.

The population's grow exponentially
while food supply increases only
arithmetically (in a linear rate)
inevitably lead to lack of resources,
famine, disease and war
In fact the balancing mechanism
is expected since due to famine
and disease, war the d parameter
is increased and starts to regulate
the size of populations.

Task 1. Find statistical data on
d and b rates during 1700-2025.
Test the validity of ODE model.

Remark 2. Try to explain how the given model can be used to simulate dynamics of ~~epidemics~~ (various infections such as SARS, plague).

3. $r < 0$

Solution $N(t) \to 0$ for $t \to \infty$.

Such a property of solutions is desired in many applications, including hospitals, pollution concentration reduction, reducing virus level).

In analysis of population dynamics it is typical ~~to~~ see the simple principal of growth ~~sets~~ must be modified by taking into account that the rate of change in the population $\frac{dN}{dt}$ is not equal to (rN) but is limited by carrying capacity $(1 - N/K)$,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \qquad (3)$$

where r is the intrisic rate of natural increase, and K is the carrying capacity.

For larger N the rate of increase of N is reduced.

This equation is called

the logistic equation.

Parameter r is called also the Malthusian parameter.

Practical tasks.

1. Find analytic solution of Ex.(3).

2. Explain why such $N(t)$ is known as the sigmoid function.

3. Solve this equation by using Matlab tools

Hint. Modify the equation (3) to equivalent equation

$$\frac{dN}{N\left(1-\frac{N}{K}\right)} = r\,dt$$

For $N = K$ this rate is equal to 0, since all resources are already used.

If $N > K$, then the size of population starts to decrease.

Next we show how the solution can be investigated by using graphic methods.

Let us start from the analysis of more general equation

$$\frac{dx(t)}{dt} = f(x). \qquad (4)$$

Note, that $f(x)$ is not depending on arguement t.

This equation defines <u>vector field</u>

$$\left(x, \frac{dx}{dt}\right) \qquad \left(\begin{array}{l}\text{we also use the} \\ \text{notation } (x, \dot{x})\end{array}\right)$$

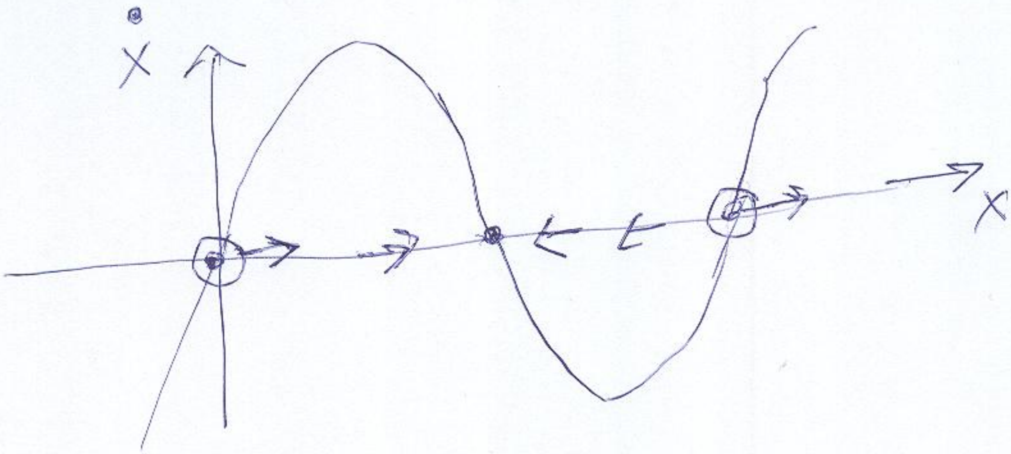
Axis x can be used to define the direction of vectors

$\dot{x} > 0$ – move to right direction

$\dot{x} < 0$ – move to left

Let us consider a simple example

$f(x) = \sin \pi x$



1. $f(x_0) = 0$, x_0 is a stationary point (fixed point)

a) • – this point is stable, $f(x)$ is decreasing here

b) ⊙ – this stationary point is not-stable ($f(x)$ is increasing)

If we select initial position x_0, then we can follow phase point $(x(t), \dot{x}(t))$ showing the movement on x-axis:

$x = x(t)$ describes a trajectory (orbit) of vector field which started at x_0.

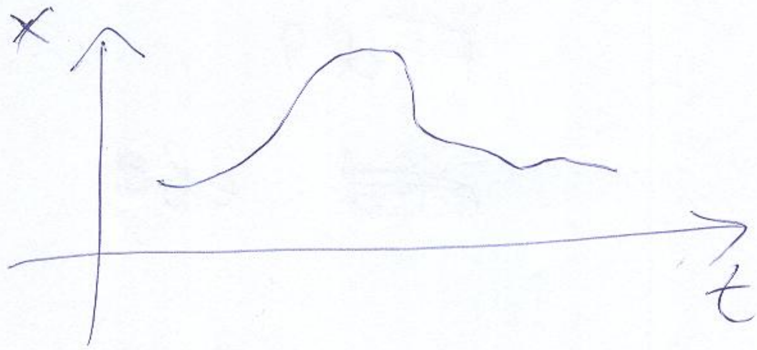


Fig.

<u>Def.</u> A phase portrait is a geometric representation of the orbits of a dynamical system in the phase plane. Each set of initial conditions is represented by a different curve. Phase portraits define an important tool in studying dynamical systems

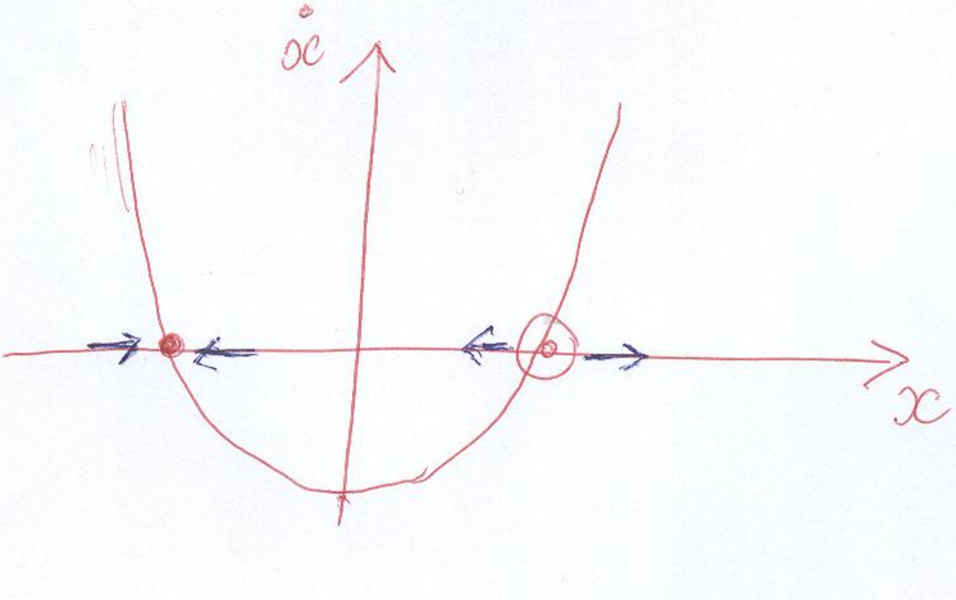
Example 1. Let us consider dynamical system, describe orbits of the solution:

$$\frac{dx}{dt} = x^2 - 1 .$$

1. First we find stationary points and make of classification of them.

The function

$f(x) := x^2 - 1$ defines parabola



We have two stationary points

$$x_1^* = -1 \quad , \quad x_2^* = 1 .$$

According to definition

$x_2^* = 1$ is a nonstable fixed point

$x_1^* = -1$ is a stable fixed point

Example 2 We consider the logistic model

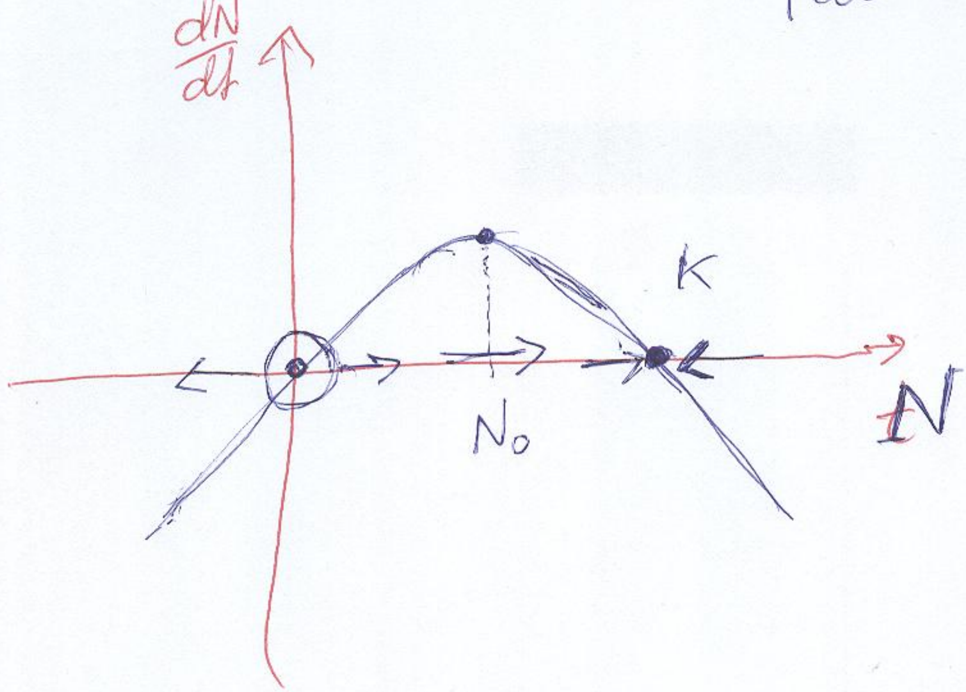
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

It follows that

$N^* = 0$ and $N^* = K$

are two stationary points.

Parabola



$N^* = K$ is a stable fixed point

$N^* = 0$ is non-stable fixed point

Task: Find the exact solution and investigate the orbits of this problem (vector fields)